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Retirement phase of superannuation – Discussion paper

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We thank you for the opportunity to respond to the discussion paper "Retirement phase of superannuation". Our response is based on our many years of experience undertaking research in the superannuation sector, including work undertaken with Aware Super, Challenger and Connexus Institute, among others. We have close ties with the sector, whilst also being able to provide and informed, arms-length and independent view.

Structure of the response

Rather than attempting to cover all areas and consultation questions of the discussion paper, we focus our response on the barriers preventing the widespread utilisation of lifetime income products (see page 23 of the discussion paper), touching briefly on other areas of the discussion paper where relevant. We use material from a research paper completed by the authors and honours student, which we include as an Appendix. We further include a bibliography of relevant papers written by the authors and colleagues.

We would be more than happy to further discuss the issues outlined in this response at your convenience.

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Lifetime Income Products in the Retirement Phase of Superannuation

Introduction

The retirement income design problem can be framed around four key building blocks, which are social security (i.e. Age Pension), survival benefits (often known as 'mortality credits' in the academic literature), investment strategies and drawdown schedules. Recent research undertaken by the authors and an honours student[1] found that sensible defaults for investment and drawdown strategies can largely replicate results from optimal strategies. However, utilisation of products with survival benefits (i.e. Lifetime Income Products, or LIPs) increases risk-adjusted retirement income by almost 20%. Hence, we believe the broader use of LIPs is the key goal to improving retirement incomes, and the remainder of this response discusses how the barriers to this goal can best be managed, splitting topics by member and provider as per the discussion paper.

A framework that aligns Lifetime Income Products with Account-Based Pensions for members

The discussion paper states that a potential barrier to LIP take-up by members is the "*upfront cost and 'wasting' capital*", noting that "*LIPs are generally sold as a one-off capital purchase in return for a lifetime income stream*". However, it is simple for a LIP to be structured like an ABP. Using t as a subscript for time, the mathematical structure is[2]:

$$A_{t+1} = A_t - D_t - P_t - F_t + I_t + S_t$$

In this structure A is the account balance, D is the drawdown/income, P are insurance premiums, F are fees, and I are investment returns. Up until this point the structure looks identical to an Account-Based Pension (ABP). The only new element of the above equation is S, which are the survival benefits of the LIP.

Under this structure, LIPs would provide annual statements to policyholders that look similar to the accumulation phase and to an ABP. And whilst a LIP will not have as much flexibility as an ABP, the similarity in reporting structure may be sufficient to lessen the barrier described above. Other points of note about this structure are:

- The essential trade-off being made in a LIP compared to an ABP is that the purchaser receives survival benefits but forfeits the right to distribute the remaining balance to their beneficiaries upon death.
- Survival benefits may be guaranteed (as is the case in most LIPs currently sold in Australia) or dependent on the mortality experience of a pool (as in the Australian Retirement Trust Lifetime Pension).
- Investment returns may be guaranteed (as in a traditional life annuity such as those sold by Challenger) or market-dependent (as in the LIPs sold by many providers).
- Fees can be explicitly incorporated into the balance calculation or may be embedded within investment returns and/or survival benefits.

[1] Butt, A., Khemka, G. and Mehry, S., 2024, A building block approach to retirement income design. *Under submission*. This paper is attached to the end of this document as an Appendix.

[2] Whilst outside the scope of the discussion paper, it is worth noting that this approach has the potential to allow improvements in consistency of Age Pension and other social security means testing, since both ABPs and LIPs would have an underlying balance at all time periods. The authors and a colleague have modelled such approaches in recent research. Ai, W., Butt, A. and Khemka, G., 2023, Interaction between Age Pension means testing and innovative income streams in Australia. *Under submission*.

- The (optional) return of capital upon death under the capital access schedule can be framed as a purchase of death insurance, with a premium hence charged to the account for this insurance.
- The (optional) return of capital upon surrender can be framed as a reduction in survival benefits, i.e. survival benefits are only paid on the balance larger than the surrender value, since the surrender value acts effectively as an ABP.
- Income from the LIP is typically a function of the balance, age and sometimes gender. This function is dependent on assumptions around future interest rates and survival benefits. The assumed interest rate (AIR) may be fixed without choice at purchase (as in a traditional life annuity such as those sold by Challenger) or may be chosen once at purchase (as in the LIPs sold by many providers).
- For a traditional life annuity, the AIR is equal to the guaranteed investment return, ensuring a level income for life. For a variable annuity, where the investment return exceeds the AIR, then income increases in the next period, and vice versa.

We have used this framing in our recent research[3] to allow consistent comparisons between various LIPs and an ABP. The AMP MyNorth Lifetime products are the only LIPs in Australia to include a balance (they call the survival benefit a 'bonus rate'). The product further allows a choice of AIR each year up to a maximum (which is also the default) with flexibility to drawdown at a lower level than this. All other LIPs in Australia do not include a balance.

Member product comparison

The discussion paper states that a potential barrier to LIP take-up by members are the "*challenges to comparison*" between products. Reframing the retirement phase to include a balance for LIPs (see above) provides greater transparency and will allow members to make easier comparisons.

The concept of a 'default' as it relates to the retirement phase is unclear, as a member cannot be defaulted into the retirement phase. Hence, even with the described reframing of the retirement phase, a straight transfer of the MySuper disclosure arrangements used in the accumulation phase is not possible. That said, we do support the need for consistency in product disclosure to facilitate comparison of products. These should be applied across both ABPs and LIPs to avoid entrenching ABPs as being 'preferred', and where possible can be applied to both the accumulation and retirement phases. Additional disclosure around flexibility, balance forfeiture and survival benefits could be incorporated for products in the retirement phase.

Using the framework above, administration fees and insurance premiums can be clearly seen in the LIP balance calculation. Like for the accumulation phase and for ABPs, some fees are embedded in elements of the balance calculation. These are discussed below:

Investment return rate and fees

For LIPs that allow policyholders to choose an investment strategy, the issue of investment fees can be treated the same as an ABP. Investment returns on periodic statements are calculated net of fees with fee levels and historical returns being disclosed in product documentation.

For a traditional life annuity guaranteeing a fixed income for life, such as those offered by Challenger, the investment return is guaranteed at the purchase of the policy within the assumptions used to calculate the income level. Whilst currently opaque, this return can be disclosed in product

[3] See footnotes 1 and 2.

documentation and calculated on periodic statements. Whilst in this case the investment fee is not explicit, a notional fee can be calculated as the difference between the returns earned by the provider on the assets backing the product and the investment return guaranteed. Returns earned by the provider on the assets backing the product are unlikely to be known in advance, and it is therefore recommended that historical fees be disclosed in product documentation.

Survival benefits and mortality rates

For LIPs that pool longevity risk, there is notionally no fee relating to survival benefit, as balances forfeited by members upon death should be fully distributed to surviving members. Calculating the survival benefits for periodic statements is achievable, and it is recommended that historical distribution rates be disclosed in product documentation.

For LIPs that are insured, the survival benefits are guaranteed at purchase within the assumptions used to calculate the income level. Whilst currently opaque, these benefits can be disclosed in product documentation, allowing comparisons with historical rates disclosed by pooled products. Further, the survival benefits can also be calculated for periodic statements. Any difference between the balances forfeited upon death and the survival benefits distributed to members represent a notional fee charged by the provider. To be consistent with investment returns, we recommend that a measure of the historical balances forfeited upon death and not distributed as survival benefits should be disclosed in product documentation.

Lifetime Income Product flexibility for members

The discussion paper states that a potential barrier to LIP take-up by members is "*lack of flexibility*". The AMP MyNorth Lifetime products show that LIPs can be developed that offer significantly more flexibility than a traditional life annuity, and as noted in the discussion paper "*most retirees would not annuitise their entire retirement savings*". However, the product does not alleviate the lack of flexibility, outside of surrender provisions, in switching providers after purchasing a LIP.

Implementation of the LIP framework above would, however, make it possible to allow switching of LIP providers, as a balance for transfer is readily available. To avoid anti-selection, there would need to be some restrictions placed on this process:

- The product being transferred into should not offer more generous surrender and/or death (including reversionary) benefit terms than the product being transferred out of.
- Restrictions would need to be placed on AIRs to ensure that products were not made available to take advantage of this option, i.e. products with extreme high AIRs would be attractive to switch into for someone of poor health. Alternatively, rules could be put in place to require balances for those who die within, say, 1 year of transferring providers, to be forfeited to the previous provider.
- LIPs with investment guarantees would need to be allowed to charge some form of disclosed exit fee to reflect the cost of unwinding the guarantee.

This proposal is rather radical, and we are not aware of it having been implemented elsewhere in the world.[4] However, it would substantially increase the flexibility of members who have purchased a LIP. This would also likely alleviate some concerns of members around "*counterparty risk*".

[4] Annuity exchanges exist in the U.S. and are known as "1035 exchanges", however this process relates to tax benefits and does not remove any surrender fees that may be charged. Most annuities sold in the U.S. include some form of embedded investment guarantee.

Lifetime Income Product development and distribution for providers

The discussion paper states that the potential barriers to LIP availability are "*development cost*", "*incentives and competition*" and "*legacy products*". Our conversations within the sector indicate that the primary barrier to the development of LIPs is concerns about the distribution of the product and particularly the risk of legacy products. A successful distribution method alleviates this concern and concerns around development cost, and may provide funds with a competitive advantage.

Hence, the implementation of the 'Delivering Better Financial Outcomes' reform is vital. As noted in the discussion paper only "*around a quarter of people seek advice as they approach retirement age retirees do not seek financial advice*", and while no data is publicly available to the best of our knowledge, anecdotal evidence suggests that most LIPs currently sold are directly or indirectly through advisors. For those not receiving advice, ABPs act as somewhat of a default, as they are closest in structure to the accumulation phase.

Allowing funds to provide more personalised advice and 'nudges' will give them the assurance to invest in LIP development, confident in this distribution method to ensure the success of the product. Furthermore, this will free funds to discharge their duties under the Retirement Income Covenant (RIC), developing 'solutions' that consider all elements of the RIC. These 'solutions' could 'wrap' together decisions about allocation to various product structures, investment strategy, incorporation of the age pension, use of home equity, and matching drawdown strategy to needed and desired expenditure. Some funds will partner with insurers in the development of such 'solutions' and other funds will deliver 'in-house'. We hope and expect to see an explosion in this space in the coming years, from funds that develop highly-specialised and bespoke solutions, to those that offer basic LIPs in addition to their current ABPs. Successful funds in the retirement phase will be those that provide an entire 'solution' to members that simplifies their financial experience in retirement, rather than just a simple ABP structure. The variety of options that we expect to be available should cater to the differing needs of retirees.

Given this, we do not believe there is need for government to intervene in providing longevity protection. Even with the concerns described above, recent collaborations between product providers and insurers (e.g. AMP and TAL) indicate market confidence in pricing the longevity risks in LIPs. Furthermore, the research presented in the Appendix finds that pooling rather than insuring this risk may increase risk-adjusted income by around 5%, assuming a sufficient pool size. This is because investment volatility even for relatively conservative portfolios tends to have a more substantial impact on outcomes than systematic mortality uncertainty, and hence the fees charged to insure longevity risk reduce its value compared to pooling longevity risk. This applies even for relatively small pool sizes of around 1,000 members[5] and thus smaller funds should be able to develop sufficiently large pools.

Our expectation of product development in this space is aspirational in nature, and hence academic investigation of the impact of the implementation of the 'Delivering Better Financial Outcomes' reforms should be encouraged. This could be followed later this decade by government assessment

[5] Qiao, C. and Sherris, M., Managing Systematic Mortality Risk With Group Self-Pooling and Annuitization Schemes. *Journal of Risk and Insurance*, 80(4), 949-974.

of the market penetration of LIPs. This should be done before stronger measures such as government-mandated default solutions or standardised/suggested products[6] are considered.

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[6] As an aside, we provide some comment here on the example product on page 28 of the discussion paper. This product incorporates an ABP combined with a deferred LIP. We agree that this example product meets the objectives of the RIC. One clear advantage of this structure is that the ABP has a fixed date to which drawdowns must satisfy. Careful choice of the settings could ensure that the ABP drawdown matches the deferred income stream. Rather than the 'capital reserve' structure, which is overly complicated and difficult to understand, providers could recommend different drawdown structures for retirees who want a consistent annual income and those who wish to set aside amounts for occasional large expenses (expected or otherwise). A further concern is that the relatively small amount which would be allocated to the deferred LIP (perhaps 10-20%) fails to unlock the full potential value of survival benefits. While we recognise that a significant portion of accumulated assets should be left to a flexible ABP, drawing down from a LIP throughout retirement will generally provide a higher standard of living, due to the additional survival benefits earned in earlier years.

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<u>Next page – Appendix (pdf version only)</u>

Butt, A., Khemka, G. & Mehry, S., 2024, A building block approach to retirement income design. Under Submission.

A building block approach to retirement income design

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Abstract

This paper addresses the retirement income planning problem from the perspective of the four main building blocks of retirement income: state pension, mortality credits, investment strategies and drawdown schedules. This unique model structure structure allows us to detail how these building blocks interact to form a retiree's overall retirement income portfolio, and what trade-offs and interactions must be considered. Under an expected utility framework, we provide a reference retiree with increasing levels of flexibility, and hence complexity, in the retirement income planning problem. We find that, under the U.S. pension system, the ability to access all building blocks increases the retiree's certainty equivalent consumption by 26.1% compared to a strategy that accesses only a private pension account with a typical balanced investment strategy and a pre-specified drawdown schedule. We find that the most substantial contributor to this increase is from utilization of the mortality credit building block (i.e. annuities).

Keywords: Retirement income, Building blocks, Mortality credits, Default design

JEL Codes: C61, D14, D15

1 Introduction

The shift away from defined benefit towards defined contribution retirement income provision leads to a transfer of longevity risk and investment risk from pension providers onto retirees. Longevity risk describes the risk that a retiree lives longer than expected and exhausts wealth that generates retirement income. Investment risk describes the uncertain nature of investment returns, and the fact that they can differ from expectations, hence affecting retirement income. Many products exist to address these risks, but are only useful if utilized by retirees. Longevity risk can be managed via products that hedge or spread longevity risk, such as annuities. Investment risk can be managed by reducing exposure to risky asset returns or through options, at the cost of lower expected returns.

The aim of retirement savings should be to generate an income (Van Wyk, 2012) and to provide income certainty (Merton, 2014) rather than wealth maximization. The disparity between the complexity of the retirement income planning problem and retirees' capacity to make optimal decisions makes it unlikely they will behave optimally. Much of the existing literature investigates the retirement income planning problem by prescribing retirees certain income products or investment strategies, or by comparing products in isolation. The limitation here lies in a lack of clarity regarding how such products can fit together to create an overall retirement income portfolio, and what interactions and trade-offs must be considered. In the event such trade-offs and interactions are addressed, results are seldom presented in a manner that conveys the main sources of income for the retiree, and the value offered by incrementally increasing flexibility in decision making.

The contribution of this paper is a unique approach to the retirement income planning problem from the perspective of income building blocks. By doing so, we explicitly convey which building blocks are most important, and hence how different retirement income products can fit together to create an optimal investment strategy and consumption profile for the retiree, given their individual characteristics. We demonstrate this approach for a representative individual with simple power utility preferences, although the approach can easily be repeated for other settings. This understanding of the impact of building blocks can help inform the design of retirement income strategies and policy.

A retiree's investment strategy and consumption profile can be attributed to four interacting building blocks: state pension, mortality credits, investment strategy, and drawdown schedule. The state pension (pension, hereafter) income is usually exogenous and can be treated as an input in the retirement planning problem (any defined benefit pension would also be a part of this building block). The remaining three building blocks represent the decisions that convert wealth saved for retirement into an income, and are integral to products available to the retirees. Each represent a dimension along which decisions must be made to achieve a preferred exposure to longevity and investment risk. Mortality credits represent a transfer of wealth from investors who die to investors who remain alive and assist in managing longevity risk. Investment strategy refers to the mix of assets in the portfolio to manage investment risk. The drawdown schedule is the amount or proportion of funds withdrawn to generate a retirement income for consumption, and interacts with both longevity and investment risk.

Linking this to products, retirees may have access to three types of products: private pension accounts (PPA), fixed and variable annuities, and group-self annuitization (GSA) or pooled mortality products. PPAs provide no mortality credits but allow for full flexibility in investment strategy and drawdown schedule as well as catering to any bequest motives. Annuities provide guaranteed (deterministic) mortality credits but generally no or only partial flexibility in the other two building blocks. GSAs offer stochastic mortality credits based on the mortality experience of the group and usually offer some flexibility in investment strategy and drawdown schedule. PPAs and GSAs are typically cheaper (i.e., have lower fees) relative to annuities since the provider of these products does not need to hold capital against longevity risk (Zhou, 2020; Weinert and Gründl, 2021). All may also have add-on features that provide optionality in investment strategy, and by extension drawdown schedule.

In this paper, we consider the retirement income problem at the time of retirement in a utility maximization framework. Figure 1 shows the overall model structure, highlighting how the different building blocks interact to create the retiree's overall investment strategy and consumption profile. At retirement, the retiree allocates their liquid wealth across three products offering different mortality credits (none, deterministic, and stochastic). Then, an investment strategy and drawdown schedule is considered for each product. Alongside pension income received, the selected investment strategies and drawdown schedules will give the retiree's overall investment strategy and consumption profile.

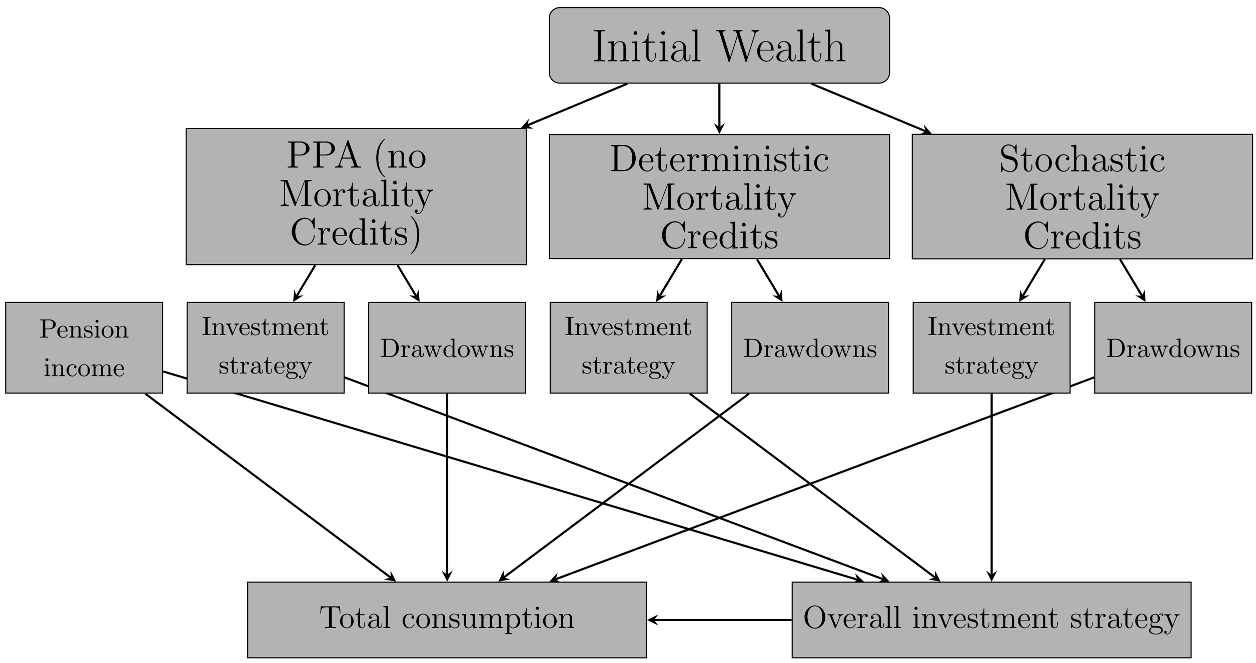

Figure 1: The interactions between the building blocks of retirement income

The existing literature on this topic typically considers only a subset of the range of building blocks and products discussed in this paper. The seminal works of Yaari (1964), Merton (1969, 1975) and Samuelson (1975) provide the foundation for the analysis of retirement consumption, wealth allocation and investment decisions when faced with an uncertain future lifetime when individuals have access to PPAs alone. MacDonald et al. (2013) review retirees' drawdown behaviour when faced with longevity and investment risk. They note that the higher flexibility in drawdowns from a PPA is one reason why they can be preferred over traditional annuity products. Milevsky (1998) finds that the higher fees in traditional annuities can be a deterrent for retirees, since they can achieve higher net returns via the PPA with a high probability until the annuity's mortality credits become materially large. Hence, there are trade-offs in allocating wealth across a PPA and a traditional annuity, with both products having attractive features which suggest some combination is optimal. Andréasson et al. (2017) analyze PPAs in the presence of the Australian means-tested pension system while Iskhakov et al. (2015), Milevsky and Huang (2018) and Butt et al. (2022) consider a combination of PPAs, pension income and fixed annuity products.

In this paper, we separate the investment strategy decision from the longevity insurance (access to mortality credits) decision. A substantial amount of existing literature does not make this separation (see for example, Davidoff et al., 2005; Lockwood, 2012; Horneff et al., 2014), partly reflecting the trade-off imposed by traditional annuities. For example, life annuities offer no exposure to risky assets, which suggests that if available alongside a PPA, they will present retirees with a trade-off between risky assets and longevity insurance. A lack of separation means the retiree's preference to insure longevity risk or access risky assets is not clear. Rather, results only indicate how retirees handle the trade-off between risky assets and longevity insurance.

Variable annuities, that allow freedom to choose the investment strategy, remove the trade-off between risky assets and longevity insurance. Steffensen and Søe (2023) suggest that these help better convey a retiree's preference for risky assets, given they no longer come at the cost of longevity insurance. Horneff et al. (2010) consider access to a PPA, a fixed pension and variable annuities and find that full annuitization is not optimal for specific drawdown schedules. Milevsky and Young (2007), conversely, find that self-annuitization of PPAs can dominate income portfolios. Ai et al. (2023) allow the retiree access to PPA, means-tested pension and variable annuities with a selectable drawdown schedule. However, none of the aforementioned literature consider a stochastic mortality credit product.

The stochastic mortality credit products commonly covered in existing literature are group self-annuitization (GSA) products, pooled annuity funds, and tontines. For our paper, the important difference between deterministic and stochastic mortality credit products is that a deterministic mortality credit product provides a complete hedge against longevity risk. Conversely, a stochastic mortality credit product spreads longevity risk across an investor pool (Zhou, 2020), thereby offering a partial hedge. We define a group-self annuitization (i.e., our chosen stochastic mortality credit product) in the same way as Piggott et al. (2005). This means it spreads longevity risk across the investor pool, and the investment risk is borne by the pool.

Stamos (2008) compares the attractiveness of GSA against PPAs and traditional annuities. However, they do not allow the retiree to spread their wealth across products. Chen et al. (2020) find that assuming reasonable loadings, a retirement income portfolio consisting of an annuity and a GSA is optimal relative to only investing in one. However, the preference across an annuity and a tontine does depend on the embedded loadings/fees, as suggested by Milevsky and Salisbury (2015).

Overall, the existing literature are subject to at least one of the following limitations. Some compare the value of different income products, but do not consider wealth allocation across products, and the trade-offs this presents the retiree. Other works do not separate the retiree's investment strategy and longevity insurance decisions. Some papers separate these decisions alongside pension income, but do not offer a stochastic mortality credit product. Overall, the literature does not approach the retirement income problem with the flexibility possible using the four building blocks, nor does it explicitly show how these building blocks and their interactions provide value during retirement.

Thus, we contribute to the existing literature by considering three distinct types of mortality credit products, and how they fit together to create an overall portfolio alongside pension income. We start with a PPA with a typical/default investment strategy and drawdown schedule, allowing the retiree to choose the optimal age for commencing the pension under the U.S. system. We then incrementally allow flexibility to the retiree across the three building blocks of mortality credits, investment strategy and drawdown schedule. Within the investment strategy, we also find the value provided to retirees when the risky asset returns are bounded through options, similar to the commonly seen variable annuity riders across all products. Our modelling approach allows us to quantify the value of each building block to the retiree as well as ascertain which building blocks add the most value to the retiree.

We find that if the retiree only has access to a private pension account (PPA) alongside their fixed pension, being able to choose an investment strategy away from the default provides more value than being able to choose a drawdown schedule. However, the reverse holds if the retiree can access products offering mortality credits, regardless of whether these mortality credits are deterministic or stochastic. Being able to use options to smooth returns provides material value similar to that of being able to choose investment and drawdown strategy. In cases where the retiree has access to products offering mortality credits and can choose their drawdown schedules, a material proportion of wealth is kept in the PPA and is used as a tool to boost consumption in the initial periods of retirement. By doing so, the retiree is able to delay receipt of and hence increase their pension income. Overall, we find that the retiree obtains most value from products offering mortality credits alongside their fixed pension income, and always allocates a majority of their wealth to these when they are available.

The rest of the paper is structured as follows. Section 2 details the methodology and assumptions underlying this paper. Section 3 provides a presentation and analysis of the results, including discussion and limitations of the analysis. Section 4 concludes the paper.

2 Methodology

2.1 Model Setup and Dynamics

We model a single male who owns their home. The individual is aged $x = 66$, has reached full retirement age (FRA) as defined in the U.S. setting (Social Security Administration, 2023b), and has fully retired and does not earn any labour income in retirement. All modelling is assumed to be in real terms to account for inflation. No tax implications of different products or cash flows are modelled.

As per the structure outlined in Figure 1, at age 66 (time, $t = 0$), the retiree allocates their wealth, W_0, across three products offering different mortality credits: a private pension account offering no mortality credits (denoted PPA), a deterministic mortality credit product (denoted DM), and a stochastic mortality credit product (denoted SM). We implement a static optimization approach under the expected utility framework, i.e., the retiree makes decisions at time $t = 0$ only, such that their expected lifetime utility is maximized. Calculations are performed in discrete annual time intervals, with t denoting the number of years after retirement and $x+T$ is the maximum age the retiree can attain. The wealth in product $i \in \{PPA, DM, SM\}$ at time $t = 0$, denoted $A_{0,i}$, is given by

$$A_{0,i} = \omega_i W_0$$

where ω_i denotes the proportion of initial wealth W_0 allocated to product i. The wealth in product i at time $t + 1$, denoted $A_{t+1,i}$, is given by the recursive equation

$$A_{t+1,i} = A_{t,i} - D_{t,i} + I_{t,i} + M_{t,i} \tag{1}$$

where $D_{t,i}$ is the drawdown from product i at time t, $I_{t,i}$ is the investment income earned by product i from time t to time $t+1$, and $M_{t,i}$ is the mortality credits earned by product i from time t to time $t+1$. The rest of this subsection describes our modelling approach for each of these three components of the recursive equation.

2.1.1 Investment returns

The retiree's investment strategy is confined to a risky asset and a risk-free asset. The risk-free asset's price process $(B_t)_{t\geq 0}$ satisfies the following differential equation

$$\frac{dB_t}{B_t} = r_f dt$$

where r_f denotes the annual risk-free rate. The risky asset's price process $(S_t)_{t\geq 0}$ is modelled as Geometric Brownian motion. Under the real-world measure $\mathbb{P}$, the risky asset satisfies the following stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$

where μ denotes the expected annual growth rate of S_t, σ denotes the annual volatility of returns for S_t, and Z_t is a Weiner process. The risky asset's realized return from time t to time $t+1$ is denoted $\hat{r}_{r,t}$.

Where the retiree has access to options, we assume that the only strategy available to the retiree is a self-financing collar strategy. In this strategy, one purchases a put option whose strike corresponds to a return of $K_{p,i}$ and sells a corresponding call option whose strike corresponds to a return of $K_{c,i}$, for product i on the risky asset such that the net cost of the strategy is zero. Let $c_t(r_f, K_{c,i}, \sigma)$ and $p_t(r_f, K_{p,i}, \sigma)$ denote the prices for the call and put options at time t which both expire in one year. With a chosen lower bound $K_{p,i}$ for the risky asset's annual returns in product i, $K_{c,i}$ is the value that satisfies the equation

$$c_t(r_f, K_{c,i}, \sigma) - p_t(r_f, K_{p,i}, \sigma) = 0. \tag{2}$$

The $K_{c,i}(> K_{p,i})$ that solves Equation (2) ensures the option trading strategy is self-financing. Both c_t and p_t are calculated using the Black-Scholes model (Black and Scholes, 1973). Then, the retiree's bounded risky asset return for product i from time t to time

$t+1$, denoted $r_{r,t,i}$, is

$$r_{r,t,i} = \hat{r}_{r,t} + \max\left(K_{p,i} - \hat{r}_{r,t}, 0\right) - \max\left(\hat{r}_{r,t} - K_{c,i}, 0\right).$$

The choice of $K_{p,i}$ (and AIR_i, see Section 2.1.3) may allow the retiree to construct a structure similar to that of a Guaranteed Minimum Withdrawal Benefit (GMWB) or other similar riders available in the marketplace.

Given the above, the investment return for product i from time t to time $t+1$, denoted as $r_{t,i}$, is

$$r_{t,i} = \pi_{r,i} r_{r,t,i} + (1 - \pi_{r,i}) r_f - f_i, \quad \pi_{r,i} \in [0, 1]$$

where $\pi_{r,i}$ is the proportion of wealth in product i invested in risky assets and f_i is the annual fee for product i. The asset allocation in each product i is rebalanced at every time t to maintain the proportion $\pi_{r,i}$ across time.

Drawdowns from each product i at time t, $D_{t,i}$, take place at the beginning of each year. Investment income is earned over each period after drawdowns occur and is credited to account balances at the end of each period. Thus, the investment income earned by product i from time t to time $t+1$ is given by

$$I_{t,i} = (A_{t,i} - D_{t,i}) \times r_{t,i}.$$

2.1.2 Mortality credits

We use the Cairns-Blake-Dowd (CBD) model proposed by Cairns et al. (2006), and mortality data from the U.S. sourced from the Human Mortality Database (2023), to project mortality rates for the purposes of calculating deterministic and stochastic mortality credits. The CBD model is briefly described below.

For all times $0 \leq t \leq T$, the probability of an individual aged $x+t$ at time t dying before age $x+t+1$, denoted Q_{x+t}, is given by the logit function

$$\ln\left(\frac{Q_{x+t}}{1 - Q_{x+t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x})$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the average of the ages used to fit the CBD model, and n is the number of distinct ages used to fit the CBD model. $\overrightarrow{\kappa_t} = \left(\kappa_t^{(1)}, \kappa_t^{(2)}\right)^\intercal$ follows a bivariate random walk with drift, such that

$$\overrightarrow{\kappa_t} = \begin{pmatrix} \kappa_t^{(1)} \\ \kappa_t^{(2)} \end{pmatrix} = \begin{pmatrix} \kappa_{t-1}^{(1)} \\ \kappa_{t-1}^{(2)} \end{pmatrix} + \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix} + \begin{pmatrix} \epsilon_t^{(1)} \\ \epsilon_t^{(2)} \end{pmatrix}$$

where $\left(\mu_t^{(1)}, \mu_t^{(2)}\right)^\intercal$ is the drift vector, $\left(\epsilon_t^{(1)}, \epsilon_t^{(2)}\right)^\intercal \sim N(0, \mathbf{\Sigma})$, and $\mathbf{\Sigma}$ denotes the variance-covariance matrix for $\overrightarrow{\kappa_t}$. The equivalent expected mortality rate, q_{x+t}, is equal to $\mathbb{E}(Q_{x+t})$, or equivalently, is found by removing the $\left(\epsilon_t^{(1)}, \epsilon_t^{(2)}\right)^\intercal$ vector in projections.

Expected mortality rates q_{x+t} are synonymous with deterministic mortality credits, and are known with certainty at $t = 0$. Actual mortality rates Q_{x+t} are synonymous with stochastic mortality credits, and are unknown at $t = 0$. We assume that the investor pool for the SM product is sufficiently large that the idiosyncratic component of longevity risk is diversified away, and is hence not modelled. The return from time t to time $t+1$ that determines the mortality credits earned by product i, denoted $\theta_{t,i}$, is

$$\theta_{t,PPA} = 0, \quad \theta_{t,DM} = \frac{q_{x+t}}{1 - q_{x+t}}, \quad \theta_{t,SM} = \frac{Q_{x+t}}{1 - Q_{x+t}}.$$

The mortality credits are earned and credited to account balances only after investment income has been credited to account balances, this gives

$$M_{t,i} = (A_{t,i} - D_{t,i} + I_{t,i}) \times \theta_{t,i}$$

The balance for each product i in Equation (1) can hence be rewritten as

$$A_{t+1,i} = (A_{t,i} - D_{t,i})(1 + r_{t,i})(1 + \theta_{t,i}).$$

2.1.3 Drawdowns

Drawdowns are made from the DM and SM products following an annuitization based drawdown strategy. We also impose the same strategy on the PPA for comparison purposes.

Let ${}_tp_x$ denote the probability of an individual at exact age x surviving to exact age $x+t$, then

$$ {}_tp_x = \prod_{s=0}^{t-1}(1 - q_{x+s}) $$

where ${}_0p_x = 1$. We denote the assumed interest rate (AIR) for product i as AIR_i. v_i is defined as the one-period discount factor corresponding to AIR_i, which is

$$ v_i = (1 + AIR_i)^{-1} $$

Let $\ddot{a}_{x+t,i}$ denote the expected present value of an immediate annuity paying \$1 annually to a retiree currently aged exactly $x + t$, with an AIR of AIR_i. The value of this annuity is

$$ \ddot{a}_{x+t,i} = \sum_{s=0}^{T-t} v_i^s \, {}_sp_{x+t} $$

The drawdown at time t from product i is given by

$$ D_{t,i} = \frac{A_{t,i}}{\ddot{a}_{x+t,i}}. \tag{3} $$

Equation (3) implies that $D_{T,i} = A_{T,i}$ for all products i. We note that, given a mortality schedule of an individual (or group), the AIR_i completely characterizes the drawdown schedule of product i.

The annuity factors $\ddot{a}_{x+t,i}$ are assumed to always be based on the expected mortality rates as at $t = 0$. Therefore, the annuity factors do not allow for systematic mortality changes. This means for the SM product, all volatility in the mortality credits is assumed to be driven by the uncertainty of Q_{x+t}, and not by repricing the annuity factors at each time t. Further, for DM products, the pricing of risks is captured through the annual fee f_i rather than differing mortality rates in drawdown calculations.

2.2 Pension Income

Pension income is assumed to be received annually at the same time as drawdowns are made. In the U.S. context, pension income received is dependent on an individual's Primary Insurance Amount, which is the amount of pension an individual receives if they choose to access pension income at their full retirement age (FRA). Additionally, individuals are rewarded for delaying receipt of their pension beyond their FRA.

Assuming a FRA of x, the benefit of delayed receipt of pension income terminates after age 70. Therefore, letting x_p denote the age at which the retiree begins receiving pension income, we have $x_p \in \{x, ..., 70\}$. Let P_{t,x_p} denote the pension income received by the retiree at time t, given they begin accessing pension income at age x_p. We find

$$P_{t,x_p} = \begin{cases} 0 & t < x_p - x \\ P_{t,x} \times (1 + 0.08 \times \min(x_p - x, 70 - x)) & t \geq x_p - x, \end{cases}$$

where $P_{t,x}$ is the pension received at FRA, which is x. Note, we assume that the individual is currently aged x and has not yet started receiving any pension income.

2.3 Utility Structure

At each time t, the retiree's drawdown from each product alongside any pension income received gives total consumption C_t as

$$C_t = D_{t,ABP} + D_{t,DM} + D_{t,SM} + P_{t,x_p}.$$

We use constant relative risk aversion (CRRA) preferences to describe the retiree's preferences over consumption (see for example, Koo et al., 2022). Under CRRA preferences, the retiree's utility from consuming C_t at time t, denoted $U_t(C_t)$, is

$$U_t(C_t) = \frac{C_t^{1-\rho}}{1-\rho}$$

where ρ describes the retiree's level of risk aversion. A higher ρ indicates a higher level of risk aversion. No bequest motive is assumed.

2.4 Optimization Procedure

Let δ denote the retiree's constant annual inter-temporal discount factor. Then, the retiree's lifetime utility, denoted V, is given by

$$V = \sum_{t=0}^{T} \delta^t \times {}_t p_x \times U_t(C_t)$$

We assume the mortality component of utility is again always based n the expected mortality rates as at t = 0. We calculate the retiree's lifetime utility over 10,000 Monte Carlo simulations (Weinert and Gründl, 2021) over multiple sets (see Section 2.6) that grant the retiree varying levels of flexibility in their decision making. Let V_n denote the retiree's lifetime utility for the n^{th} Monte Carlo simulation, then the objective function to optimize is

$$V^\star = \max_{\omega_i \in \Omega,\ \pi_{r,i} \in \Pi,\ AIR_i \in AIR,\ K_{p,i} \in K, x_p \in X_p} \left(\frac{\sum_{n=1}^{10000} V_n}{10000} \right) \tag{4}$$

for $i \in \{PPA, DM, SM\}$, subject to the following constraints

$$\begin{aligned}
\Omega &= \{(\omega_{PPA}, \omega_{DM}, \omega_{SM}) | 0 \leq \omega_{PPA}, \omega_{DM}, \omega_{SM} \leq 1, \omega_{PPA} + \omega_{DM} + \omega_{SM} = 1\} \\
\Pi &= \{(\pi_{r,PPA}, \pi_{r,DM}, \pi_{r,SM}) | 0 \leq \pi_{r,PPA}, \pi_{r,DM}, \pi_{r,SM} \leq 1\} \\
AIR &= \{(AIR_{PPA}, AIR_{DM}, AIR_{SM}) | AIR_{PPA}, AIR_{DM}, AIR_{SM} > -1\} \\
K &= \{(K_{p,PPA}, K_{p,DM}, K_{p,SM}) | -1 \leq K_{p,PPA}, K_{p,DM}, K_{p,SM} \leq r_f\} \\
X_p &= \{66, 67, 68, 69, 70\}.
\end{aligned}$$

where $V^\star$ is the maximized expected lifetime utility for a given set.

2.5 Comparison Metric

We calculate a certainty equivalent consumption (CEC) for the retiree's maximized expected lifetime utility, $V^\star$, under each set as

$$V^\star = \sum_{t=0}^{T} \delta^t \times {}_t p_x \times \left(\frac{CEC^{1-\rho}}{1-\rho} \right)$$

The CEC reflects the constant level of consumption in retirement which gives the retiree a utility equal to their maximized expected lifetime utility. The changes in CEC across different sets indicate the value provided to the retiree by varying levels of flexibility in building blocks available.

2.6 Analysis Structure

The pension building block and choice of age X_p is assumed to be available in all sets. In order to align building blocks with product availability, we treat the mortality credit building block (PPA, DM and SM) as being the product spaces. The investment strategy and drawdown schedule building blocks are called decision sets. The sets over which optimization takes place make up the intersection of the decision sets and product spaces. Table 1 describes the five decision sets considered which grant the retiree different levels of flexibility in decision making.

Table 1: Decision sets

Decision Set	**Optimization variables**
1	Ω, X_p
2	Ω, X_p, Π
3	Ω, X_p, AIR
4	Ω, X_p, Π, AIR
5	Ω, X_p, Π, AIR, K

Decision Set 1 provides a baseline for comparison. Set 2 shows the isolated impact of allowing choice of risky asset allocation. Set 3 shows the isolated impact of allowing choice of drawdown schedule. Set 4 shows the interactive impact of Set 2 and Set 3. Set 5 shows the impact of using options to bound the risky asset return. Within each Decision Set, there are three product subsets, as described in Table 2, which show the impact of allowing different mortality credit structures.

Table 2: Product subsets for optimization

Product Subset	**PPA product**	**DM product**	**SM product**
*.1	✓		
*.2	✓	✓	
*.3	✓	✓	✓

The combination of a decision set and product subset produces what we call a ‘set’. In each decision set above, the vectors Ω, Π, K, and AIR are adjusted as necessary to reflect the available decision variables within that set. For example, in Set 5.2 the SM product is unavailable but all investment strategy and drawdown schedule building blocks can be freely chosen, so Ω represents $(\omega_{PPA}, \omega_{DM})$, Π represents $(\pi_{r,PPA}, \pi_{r,DM})$, AIR represents (AIR_{PPA}, AIR_{DM}) and K represents $(K_{p,PPA}, K_{p,DM})$.

In some sets some optimization variables may not be freely chosen. For example, in Set 2.2 the allocation to risky assets in both PPA and DM may be chosen, but options are not available and the drawdown schedule is fixed. See Section 2.8 for discussion of the default value for variables that are not freely chosen.

2.7 Parameter Calibration

Table 3: Parameter values

Parameter	Description	Value	Reference
x	Age at retirement	66	(Social Security Administration, 2023b)
T	Maximum years lived in retirement	38	Section 2.1.2
ρ	CRRA risk-aversion	4	(Khemka et al., 2021)
δ	Intertemporal discount factor	0.977	(Kudrna et al., 2022)
μ	Expected risky asset return	0.065	(Dimson et al., 2023)
σ	Risky asset volatility	0.174	(Dimson et al., 2023)
r_f	Risk-free rate	0.015	(Dimson et al., 2023)
f_{PPA}	PPA fee	0	
f_{DM}	DM fee	0.01	(Ganegoda and Bateman, 2008)
f_{SM}	SM fee	0	
$P_{t,66}$	Pension at FRA	\$21,552	(Social Security Administration, 2023a)
CEC_0	Initial CRRA certainty equivalent consumption	\$50,000	(Guzman and Kollar, 2023); (Whitehouse and Queisser, 2007)
W_0	Initial wealth	\$617,762	Section 2.8

Table 3 states the parameter values used along with their references. The parameters without stated references are assumed.

We assume $T = 38$ because sufficient mortality data is not available for ages beyond 104.

The DM product's annual fee f_{DM} captures the product provider's requirement to hold capital against longevity risk. The fee of $f_{DM} = 1\%$ per annum is equivalent to an approximately 12% initial loading, which is broadly consistent with the findings of Ganegoda and Bateman (2008). Neither the PPA nor the SM product require capital to be held against longevity risk. Furthermore, other fees common across the three products are assumed to be zero. As described by Ai et al. (2023), we make this assumption because only the relative value of the fees matters for comparison purposes.

The use of $P_{t,66} = \$21,552$ is consistent with the monthly pension data provided by the Social Security Administration (2023a), which is first rounded down to the nearest dollar (Social Security Administration, 2003).

2.8 Initialization Procedure

This section outlines the initialization procedure used before optimization to establish the value of W_0. For all Sets except Set 5.3, some of the variables in Equation (4) are set to default values rather than optimized. Using the superscript def to denote a default variable, the structure is as follows.

$$\begin{aligned} \pi_{r,i}^{def} &= 0.5 \\ AIR_i^{def} &= \pi_{r,i}\mu + (1-\pi_{r,i})r_f - f_i - \frac{1}{2}(\pi_{r,i}\sigma)^2 \\ K_{p,i}^{def} &= -1 \end{aligned}$$

The use of $\pi_{r,i}^{def} = 0.5$ as the default investment strategy is appropriate for our chosen $\rho = 4$, following Khemka et al. (2021) and is similar to the Vanguard Target Retirement Fund in the US (Morningstar, 2018). The form of AIR_i^{def} matches the geometric average return of the default investment strategy (less any fees), given it reflects the annualized average return in a deterministic projection (McCulloch, 2003). This gives an expected flat drawdown profile for a chosen investment strategy. The choice of $K_{p,i}^{def} = -1$ ensures that options have no impact on returns.

We set W_0 to be that required for the retiree to achieve a $CEC = \$50,000$ for Set 1.1. In Set 1.1, the only decision variable for the retiree is age to access pension, x_p, and so in addition to above, we impose $\omega_{PPA}^{def} = 1$. \$50,000 is chosen as an approximation of applying a 70% replacement rate (Whitehouse and Queisser, 2007) to the median income for males with no spouse in the U.S., which was \$73,630 in 2022 (Guzman and Kollar, 2023). The retiree's objective function in Set 1.1 is therefore

$$\max_{x_p \in \{66,67,68,69,70\}} \left(\frac{\sum_{n=1}^{10000} V_n}{10000} \middle| \omega_{PPA}^{def}, \pi_{r,PPA}^{def}, AIR_{PPA}^{def}, K_{p,i}^{def} \right) \tag{1}$$

The initialization procedure yields $W_0 = \$617,762$ with the retiree choosing to optimally begin accessing pension income at age $x_p = 68$.

3 Results

The optimization results are presented in Table 4. Following this, the analysis of results is partitioned based on the level of flexibility provided to the retiree in their decision making. Sections 3.1-3.5 analyze the results for Sets 1-5 respectively. For some sets, we use graphical illustrations of the median consumption profiles, attributing consumption to its component building blocks. The components are pension, capital drawdowns from the initial wealth (account balance), drawdowns from investment income (net of fees) earned since retirement, and drawdowns from mortality credits earned since retirement. Details on the construction of the graphs is provided in Appendix A[1], including all plots for all sets. We conclude this section with a brief discussion of the results in Section 3.6.

3.1 No decision variables

In Set 1.1, the retiree only has access to a PPA which does not provide any mortality credits, and the investment and drawdown strategies are set to their defaults. They delay pension receipt by two years to age $x_p = 68$ which increases their pension income to $P_{t,68} = \$24,996$ from $P_{t,66} = \$21,552$. Since the retiree has a fixed drawdown schedule, this reduces their consumption in the first two years of retirement, which the retiree is willing to endure for a higher consumption safety net.

With access to a DM product in Set 1.2, the retiree allocates $\omega_{DM} = 100\%$ of their wealth to the DM product, consequently increasing their CEC by 13.2% relative to Set 1.1. This conveys the value provided by deterministic mortality credits to the retiree, despite the annual fee of $f_{DM} = 1\%$. The retiree also opts to receive pension income from age 66, since the DM product offers the retiree longevity insurance, thereby reducing the incentive to delay pension income receipt to obtain a higher pension. In Set 1.1, the retiree faces a trade-off between adequate longevity insurance and initial consumption.

[1]In these plots, allocations to products $< 1\%$ of wealth are excluded. Noting that such allocations are always accompanied with an extremely negative AIR, this effectively creates a deferred annuity that finances consumption materially only in the last years of retirement. Graphing such results created consumption plots whose scales were significantly impacted by consumption at time $T = 38$.

Table 4: Fixed pension optimization results

Set	ω_{PPA}	ω_{DM}	ω_{SM}	$\pi_{r,PPA}$	$\pi_{r,DM}$	$\pi_{r,SM}$	$K_{p,PPA}$	$K_{p,DM}$	$K_{p,SM}$	AIR_{PPA}	AIR_{DM}	AIR_{SM}	x_p	$\%\Delta CEC$ from Set 1.1	$\%\Delta CEC$ from previous set
Decision Set 1 —No decision variables (Section 3.1)															
1.1	*100.0%*			*50.0%*						*3.6%*			68		
1.2	0.0%	100.0%		—	*50.0%*					—	*2.6%*		66	13.2%	13.2%
1.3	0.0%	0.0%	100.0%	—	—	*50.0%*				—	—	*3.6%*	66	19.7%	5.8%
Decision Set 2 —Access to investment strategy (Section 3.2)															
2.1	*100.0%*			94.4%						*4.9%*			69	3.1%	
2.2	0.0%	100.0%		—	68.9%					—	*3.2%*		66	14.1%	10.7%
2.3	0.0%	0.0%	100.0%	—	—	64.9%				—	—	*4.1%*	66	20.4%	5.4%
Decision Set 3 —Access to drawdown schedule (Section 3.3)															
3.1	*100.0%*			*50.0%*						6.1%			70	1.7%	
3.2	12.1%	87.8%		*50.0%*	*50.0%*					73.8%	2.1%		68	14.3%	12.4%
3.3	11.3%	0.2%	88.5%	*50.0%*	*50.0%*	*50.0%*				82.1%	-38.4%	2.7%	68	20.7%	5.6%
Decision Set 4 —Access to both investment strategy and drawdown schedule (Section 3.4)															
4.1	*100.0%*			89.4%						6.7%			70	3.9%	
4.2	21.3%	78.7%		0.4%	86.0%					40.7%	2.3%		69	16.0%	11.6%
4.3	11.4%	0.2%	88.4%	0.0%	96.7%	76.6%				81.2%	-39.0%	3.0%	68	22.2%	5.4%
Decision Set 5 —Access to options in the investment strategy and including drawdown schedule (Section 3.5)															
5.1	*100.0%*			100.0%			-17.7%			6.6%			70	6.5%	
5.2	12.1%	87.9%		99.9%	100.0%		-3.4%	-16.0%		75.1%	2.7%		68	19.6%	12.3%
5.3	11.2%	0.2%	88.6%	97.8%	99.6%	100.0%	-2.5%	-17.8%	-15.6%	83.7%	-38.1%	3.3%	68	26.1%	3.2%

Note, the gray shaded blocks represent decisions that were imposed at default values in a set (see Section 2.8). Cells with '—' represent decisions that were not used as the allocation to the product was 0%.

This trade-off is dampened in Set 1.2 as deterministic mortality credits are available to the retiree.

In Set 1.3, the retiree allocates $\omega_{SM} = 100\%$ of their wealth to the SM product, leading to a further 5.8% increase in CEC from Set 1.2. This improvement is attributable to the lack of fees in the SM product though the effect is dampened due to the uncertain nature of its mortality credits in the SM product. The pension access age remains at 66, and we note that the retiree is willing to absorb the volatility of the SM product's payouts in exchange for its 'free' mortality credits.

Figure 2 shows the attributed median consumption profiles for Set 1. The different subsets show the overall shape of the retiree's median consumption profile and its changes as different mortality credit structures are accessible to the retiree.

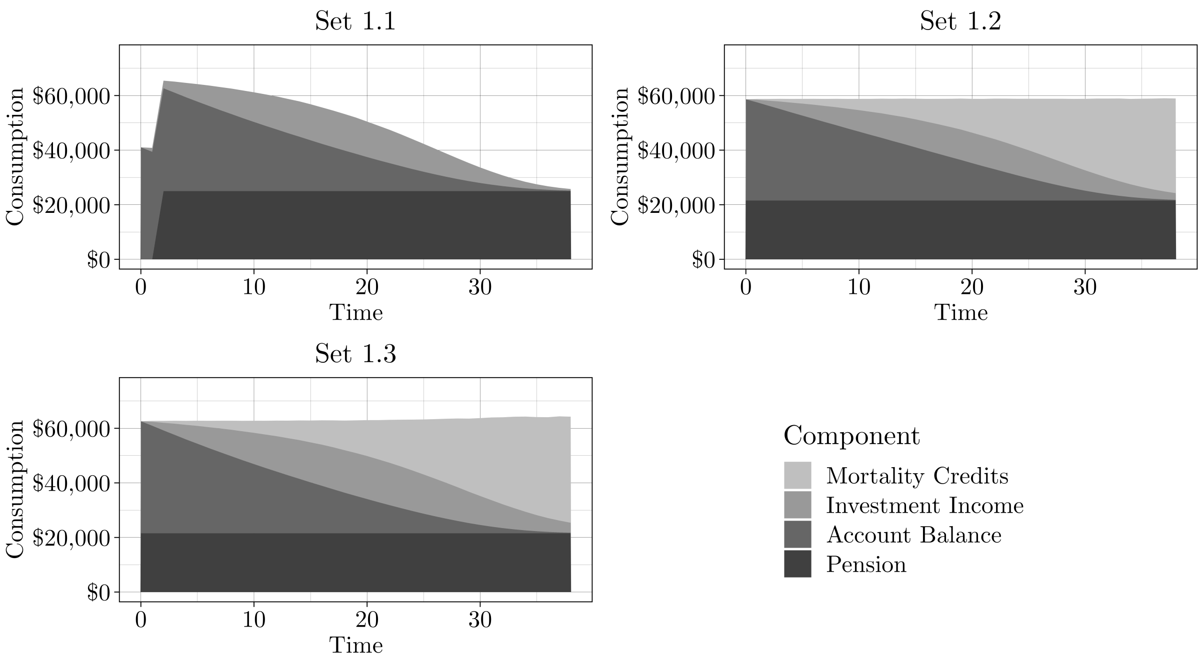

Figure 2: Median consumption profiles under Decision Set 1

Set 1.1 displays a decreasing consumption profile, attributable to the lack of mortality credits provided by the PPA, which incentivizes the retiree to delay pension income receipt until age $x_p = 68$ in order to achieve a sufficiently high level of consumption in later years. Consumption declines because the PPA's lack of mortality credits means that drawing down in accordance with its geometric mean return is not sustainable.

In Set 1.2, the consumption level is flat indicating the stable payouts from the pension and (annuity-like) DM product. Here, the shape of the drawdowns from account balance and investment returns are identical to the PPA product, but the stability in later years is achieved through the mortality credits available to the retiree. In Set 1.3, the pattern is similar to Set 1.2, but at a slightly higher level due to the lack of fees. This higher level is reflected in the higher "Investment Income" component, as the DM fee is included in the net investment return.

3.2 Access to investment strategy

In Set 2 we see the improvements in the retiree's lifetime expected utility when they can set the risky asset allocation for each product. The shapes of the median consumption profiles are similar to those presented in Figure 2 and are presented in Appendix A.

In Set 2.1, we see that the retiree chooses a risky asset allocation of $\pi_{r,PPA} = 94.4\%$ in their PPA. This improves their CEC by 3.1% from Set 1.1 where the risky asset allocation was fixed at $\pi^{def}_{r,PPA} = 50\%$. This higher risky asset allocation is driven by the retiree's receipt of fixed pension, which incentivizes them to increase their exposure to risky assets in hopes of higher returns, knowing they have a consumption safety net. This strategy also leads to an increase in the default AIR_{PPA} to 4.9% meaning their initial drawdowns from the PPA are higher relative to Set 1.1. Therefore, the retiree is willing to delay pension receipt even more than in Set 1.1, to age $x_p = 69$.

In Set 2.2, the retiree allocates $\omega_{DM} = 100\%$ of their wealth to the DM product, with an increase in CEC of 10.7% from Set 2.1. The increase is a modest 0.8% compared to Set 1.2, indicating relatively little improvement from choosing investment strategy. This is despite the change in risky asset allocation from 50% to 69%, indicating that even moderate changes in risky asset allocation away from 50% have minimal impact on outcomes.

Set 2.3 results largely mirror the findings from Set 1.3 and Set 2.2 on the impact of SM and choice of risky asset allocation.

3.3 Access to drawdown schedule

Set 3 allows the retiree to choose an AIR, i.e., drawdown schedule, but the investment strategy is set at the default as per Set 1. The median consumption profiles for this set are presented in Figure 3.

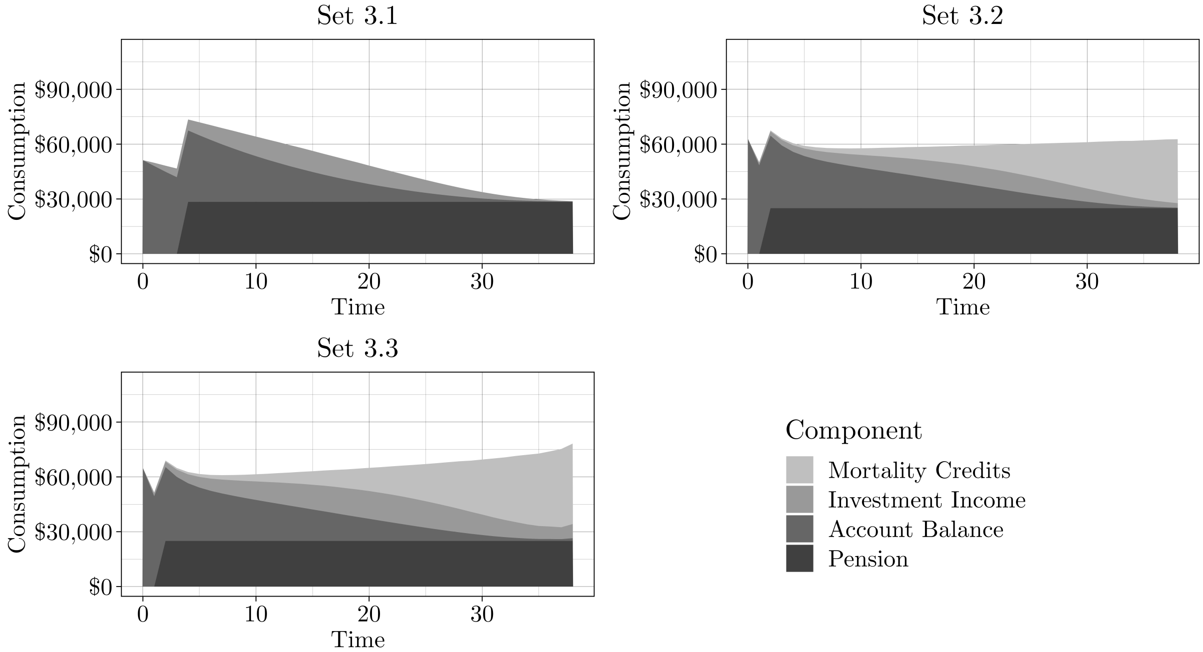

Figure 3: Median consumption profiles under decision set 3

In Set 3.1, we see an increase in CEC of 1.7% compared to Set 1.1., driven by the retiree choosing to increase AIR_{PPA} to 6.1%. The retiree also chooses to access the maximum pension possible of $P_{t,70} = \$28,452$ by drawing from age $x_p = 70$. The retiree essentially draws down on PPA faster in the earlier years to delay and increase pension receipt. From the top-left panel of Figure 3 we see that the shape is similar to the corresponding panel of Figure 2 but at a higher level of consumption throughout. We note that for the PPA account, access to investment strategy is of more benefit than drawdown schedule, in isolation.

Unlike Sets 1.2 and 2.2, in Set 3.2, the retiree maintains an allocation to the PPA of $\omega_{PPA} = 12.1\%$ and delays pension receipt to age $x_p = 68$. The retiree funds consumption in the early years from the PPA as reflected through $AIR_{PPA} = 73.8\%$. This leads to a 12.4% increase in CEC above Set 3.1, which also means Set 3.2 provides slightly greater value beyond Set 1.1 relative to Sets 1.2 and 2.2. In other words, choice of drawdown strategy provides less value than choice of risky asset allocation to the retiree when only a PPA is available, but more value when DM is available as well. From the top-right panel of Figure 3 we can see that the high AIR of the PPA account leads to a dip in consumption at age 67 as very little funds are left in the PPA account by the second year, but this is more than compensated with pension receipt at age 68. Further the low $AIR_{DM} = 2.1\%$ leads to an increasing overall consumption profile as the retiree ages.

Set 3.3 again largely mirrors the findings from Set 1.3 and Set 3.2 on the impact of SM and choice of AIR. The $AIR_{SM} = 2.7\%$ is higher than the $AIR_{DM} = 2.1\%$ in Set 3.2, although the difference is smaller than the DM fee $f_{DM} = 1\%$ and so the bottom-left panel of Figure 3 shows a stronger increasing consumption profile than Set 3.2. The small $\omega_{DM} = 0.2\%$ allocation to the DM structure, coupled with its $AIR_{DM} = -38.4\%$, suggests it is mostly being used to increase consumption in the final few years of retirement (like a deferred annuity).

3.4 Access to both investment strategy and drawdown schedule

In Set 4, the retiree can choose both investment strategy and drawdown schedule. The shapes of the median consumption profiles are similar to those presented in Figure 3 and are presented in Appendix A.

Rather than commenting on each set individually, we note that the retirees in Set 4 utilise a combination of the strategies observed in Set 2 and Set 3. In all cases, the improvement in CEC is greater than the equivalents in Set 2 and 3, as expected.

We are now also able to comment on the relative impact of the various building blocks. Clearly the addition of mortality credits has the biggest individual impact on CEC, with increases of 13.2% and 19.7% for DM and SM in Sets 1.2 and 1.3, respectively. Whereas the individual addition of investment strategy and drawdown schedule decisions leads to CEC improvements of only 3.1% and 1.7%, in Sets 2.1 and 3.1, respectively. Collectively, the impact of investment strategy and drawdown schedule improves CEC by only 3.9% in Set 4.1. This relative impact is despite the presence of the pension, which provides some longevity protection on its own. Including investment strategy and drawdown schedule decisions in addition to access to SM improves CEC by 22.2% in Set 4.3, compared to 19.7% in Set 1.3 where investment strategy and drawdown schedules are imposed at the default. This represents only a 2.1% improvement in CEC when allowing a choice in investment strategy and drawdown schedules when SM is available.

3.5 Access to options in the investment strategy and including drawdown schedule

In Set 5, the retiree has access to options in the investment strategy, in addition to the choice of risky asset allocation and drawdown schedule. The shapes of the median consumption profiles are similar to those presented in Figure 3 and are presented in Appendix A.

When the retiree only has access to PPA (Set 5.1), they allocate 100% of the wealth to the risky asset (up from 89% in set 4.1) but truncate the return distribution to $(-17.7\%, 27.5\%)$ through the use of options. This truncation allows the retiree to minimise the impact of severely negative risky asset returns and increases CEC by 2.5% compared to Set 4.1. This is a substantial outcome given that the CEC impact of choosing risky asset allocation in Set 2.1 is 3.1%.

In Set 5.2, the retiree's allocation across PPA and DM are identical to Set 3.2 as is the pension access age of $x_p = 68$. However, the retiree allocates almost 100% of the wealth in both products to the risky asset. They truncate the PPA return distribution to $(-3.4\%, 7.3\%)$ to ensure funds to finance early consumption. The DM return distribution is much wider at $(-16.0\%, 24.9\%)$. The ability to choose investment strategy and an appropriate return range increases CEC by 4.7% relative to Set 3.2 and by 3.2% relative to Set 4.2. The choice of truncation for the DM return distribution indicates that it is not optimal for the retiree to seek a GMWB structure, as the lower bound of -16.1% provides ample opportunity for the income from the DM product to decrease between years, even accounting for the fact that we are modelling in real rather than nominal terms.

In Set 5.3, the retiree has access to all products (including options), and to all decision variables. Decisions are broadly similar to those noted for Set 5.2. The retiree's CEC increases by 3.2% above Set 4.3, compared to the increase of 2.5% for Set 5.1 above Set 4.1 noted earlier, indicating that the use of options provides greater value when mortality credits are available than if they are not. Furthermore, the increase of 3.2% is greater than the increase of 2.1% observed when including both risky asset and drawdown decisions, indicating that the use of options improves outcomes more than both of these decisions.

3.6 Discussion

In this subsection, we further discuss the optimization results presented earlier in this Section, emphasising the implications of our results. We also discuss some additional robustness tests performed and address the key limitations of our analysis.

Our analysis allows for an explicit identification of the value provided by each building block of retirement income to the retiree. This also delivers insights into the interactions between the building blocks, and how the retiree behaves in light of this. While mortality credits provide the most value, this result is also driven by the fact that all products had a default investment strategy balanced between risky and risk-free assets. That is, when mortality credits are offered to the retiree, in the most restrictive settings they came with a 50% allocation to risky assets. As discussed in Section 2.8, this default is based on the literature and defaults available in the marketplace. Had we instead defined the default investment strategy as being risk-free, then the value added by allowing risky investment (whilst still imposing the drawdown schedule) would be 14.7% with PPA only (i.e. Set 2.1), and 11.4% with PPA, DM and SM (i.e. Set 2.3). This improvement is closer to, but still below the 19.7% improvement noted in Section 3.4 for the use of SM in isolation.

In sets where the retiree is not able to select an AIR for the products, we see a clear dominance by the SM product, and the DM product (in sets where the SM product was not available). The dominance of the SM product stems from lower fees relative to the DM product. Upon further investigation, we find that the annual DM fee f_{SM} must be no more than 0.1% (compared to the selected value of 1%) for the DM product to be competitive with the SM product. This implies that the retiree's 'value' of the greater stability of the DM product is less than 0.1% per annum. Part of the reason for this low value is likely due to volatility in the SM product being overwhelmed by the larger scale of investment volatility. We note, however, the simplistic approach used in this paper to model the SM mortality credits, and leave for future research to incorporate idiosyncratic volatility and repricing of the SM product into the model (see for example Qiao and Sherris, 2013).

The dominance of SM and DM products over the PPA holds partly due to the lack of a bequest motive in our analysis. We are assuming a retiree whose aim is to provide an income in retirement, ignoring any goals for inter-generational wealth transfer and/or contingent asset provision. Consideration of these motives in this setting will likely lead to different trade-offs and interactions and is left as an area for future research.

In sets where the AIR of products could be chosen, the retiree uses the PPA as an immediate consumption tool. By doing so, the retiree is willing to delay pension income receipt, thereby increasing the pension they eventually receive. This highlights the interaction between the ability to choose the drawdown schedule of the PPA, with the decision to delay receipt of pension income in order to receive a higher pension income. This result is observed despite the model limitation of static optimization at time $t = 0$ instead of a more comprehensive dynamic programming method. The inflexibility of the static optimization approach leads to large changes in median consumption in early retirement, as the use of the PPA transitions to the commencement of the pension. The static approach was chosen given the complexity of the problem addressed. By Set 5.3, the retiree has 12 decision variables to consider. We hypothesize that the impact of the drawdown decision would be substantially greater were a dynamic model possible. Further, a byproduct of the static structure is that optimal investment strategy decisions at time $t = 0$ can drift over time. For example, if the retiree implements different investment strategies for each product at time $t = 0$, even if the investment strategy is rebalanced each year within each product, their overall investment strategy will drift according to changes in the relative size of each account balance. Hence, we also hypothesize that the impact of the investment strategy decision would also be greater in a dynamic model.

This paper demonstrates that differences in optimal allocation to risky assets and use of options is significantly different across the products, which we can see as our model explicitly separates the investment strategy and longevity insurance decisions. The use of options does not replicate the riders widely available in the marketplace, instead seeking to avoid only the most severe market downturns, rather than providing a guarantee on returns. This result is dependent on the constant relative risk aversion utility structure used in the analysis, we leave to future research to investigate the impact of alternative preference structures such as cumulative prospect theory (Tversky and Kahneman, 1992), habit formation (Pollak, 1976) and Epstein-Zin (Epstein and Zin, 2013).

We now turn to the implications of our work. Our results show that sensible default investment strategies and drawdown schedules can largely replicate optimal outcomes.

Whilst the use of options can add some value, their relative complexity means they will likely be of interest to only a small cohort of retirees. Mortality credits are the most significant generator of added value in retirement, however we note that in 2017, only 7.9% of retirees received any form of annuity income (Congressional Research Service, 2022). This 'annuity puzzle' has been widely researched (Ramsay and Oguledo, 2018), however our research shows the importance of extending access to, and utilization of, products including mortality credits, particular those that don't incorporate significant fees. Providers and policymakers should be cognisant of this outcome in seeking to improve the market penetration of these products. As part of this goal, we note that the fee charged for the DM product is transparent in our model, whilst in practice transparency of fees for these products is often problematic and can distort the true value of the product.

In addition to those described earlier in this section, other simplifications were made in the analysis, and could be relaxed for future research. This could include analysis of couples in addition to singles. Tax and minimum drawdown rules could also be incorporated. Additionally, no allowances were made for increasing healthcare needs, or the need for aged care, as the retiree ages. These can be incorporated via utility functions that incorporate a non-constant consumption target (which can be incorporated under prospect theory). Housing wealth, and the ability to use a reverse mortgage to further increase consumption was also not considered. Including these will likely require consideration of bequest motives, as discussed earlier. Further, these alternatives may lend themselves to alternative defaults to those used in this research.

4 Conclusion

We explore the retirement income problem from the perspective of four interacting building blocks. These four building blocks are pension income, mortality credits, the investment strategy and the drawdown schedule. The modelled retiree is assigned the task of maximising their expected lifetime utility by considering the decision variables they have available at the time of retirement, which pertain to the aforementioned building blocks.

Over the course of many sets of decision variables, the flexibility of the retirement income problem increases, thereby reducing its parsimony. This is done by offering the retiree more mortality credit products to invest in, and more accompanying investment and drawdown decisions to make. The retiree's utility under different decision variable sets is then compared using a certainty equivalent consumption (CEC) for each set. We consider the problem primarily under the U.S. fixed pension system[2]. Utilising this breakdown of the retirement income problem, we achieve an understanding of the dynamics and interactions between these building blocks, and what trade-offs the retiree faces. The main contribution of this paper to existing literature is its unique approach to the retirement income problem from the perspective of these four building blocks, which allows for a quantification of the value provided by each. For our model retiree, we find that utilization of mortality credits provides the largest increase in welfare of the availability building blocks and that, given sensible defaults, investment strategy and drawdown schedule offer far less scope for welfare improvement.

We acknowledge that the retirement income problem is not simple. For most retirees, the simplest, and subsequently default option, is keeping wealth in a private pension account and making drawdowns as necessary. However, this reduces the overall income delivered during retirement. Through its unique representation of the retirement income problem, this paper shows that spreading wealth across products, some of which offer longevity insurance via mortality credits, can greatly improve retirement outcomes. The approach used in this paper can be utilised across other retiree characteristics and preferences, and can be utilized by providers and policymaker to assist in the retirement planning problem across cohorts.

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[2]The authors also perform sensitivity analyses for a means-tested pension and no pension. These are available upon request.

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A Graphical results

This appendix describes the graphical representation of the retiree's median consumption profile, along with presenting the plot for each set. Note that in all graphs, allocations to products $< 1\%$ of wealth were excluded (see footnote 1).

Table 5 defines the necessary notation to explain the underlying mathematical formulation of the graphs.

Table 5: Graph generation notation

Term	**Definition**
$\text{geo}(r_{t,i})$	Geometric average investment return for product i from time t to $t+1$ across all 10,000 simulations
$\text{med}(A_{t,i})$	Median account balance for product i at time t across all 10,000 simulations
$\text{med}(C_t)$	Median total consumption at time t across all 10,000 simulations
$A_{UAB,t,i}$	Underlying account balance for product i at time t
$A_{IIB,t}, i$	Investment income balance for product i at time t
$A_{MCB,t,i}$	Mortality credits balance for product i at time t
$D'_{UAB,t,i}$	Amount of drawdown from product i at time t attributable to underlying account balance
$D'_{IIB,t,i}$	Amount of drawdown from product i at time t attributable to investment income balance
$D'_{MCB,t,i}$	Amount of drawdown from product i at time t attributable to mortality credits balance

For product $i \in \{DM, SM\}$, at every t we partition its account balance into 3 components: the underlying account balance ($A_{UAB,t,i}$), the investment income balance ($A_{IIB,t,i}$), and the mortality credits balance ($A_{MCB,t,i}$). Then, the values of $A_{UAB,t+1,i}, A_{IIB,t+1,i}$, and $A_{MCB,t+1,i}$ for $t \geq 0$ are found using

$$A_{UAB,t+1,i} = \text{med}(A_{t,i}) - \frac{\text{med}(A_{t,i})}{\ddot{a}_{x+t,i}} \tag{A.1}$$

$$A_{IIB,t+1,i} = \left(\text{med}(A_{t,i}) - \frac{\text{med}(A_{t,i})}{\ddot{a}_{x+t,i}}\right) \times \text{geo}(r_{t,i}) \tag{A.2}$$

$$A_{MCB,t+1,i} = \left(\text{med}(A_{t,i}) - \frac{\text{med}(A_{t,i})}{\ddot{a}_{x+t,i}}\right)(1 + \text{geo}(r_{t,i}))\left(\frac{q_{x+t}}{1 - q_{x+t}}\right) \tag{A.3}$$

where $\ddot{a}_{x+t,i}$ is defined in Section 2.1.3 and $A_{MCB,T,i} = 0$. The use of q_{x+t} rather than Q_{x+t} for the SM product is because the median stochastic mortality rates are equal to the deterministic mortality rates.

For product $i \in \{DM, SM\}$, at each time t, the component of its drawdown $\frac{\text{med}(A_{t,i})}{\ddot{a}_{x+t,i}}$ attributable to $D'_{UAB,t,i}, D'_{IIB,t,i}$ and $D'_{MCB,t,i}$ is given by

$$D'_{UAB,t,i} = \left(\frac{A_{UAB,t,i}}{\ddot{a}_{x+t,i}}\right)\left(\frac{\text{med}(A_{t,i})}{A_{UAB,t,i} + A_{IIB,t,i} + A_{MCB,t,i}}\right) \tag{A.4}$$

$$D'_{IIB,t,i} = \left(\frac{A_{IIB,t,i}}{\ddot{a}_{x+t,i}}\right)\left(\frac{\text{med}(A_{t,i})}{A_{UAB,t,i} + A_{IIB,t,i} + A_{MCB,t,i}}\right) \tag{A.5}$$

$$D'_{MCB,t,i} = \left(\frac{A_{MCB,t,i}}{\ddot{a}_{x+t,i}}\right)\left(\frac{\text{med}(A_{t,i})}{A_{UAB,t,i} + A_{IIB,t,i} + A_{MCB,t,i}}\right) \tag{A.6}$$

respectively. The expressions above involve a scaling of $(\text{med}(A_{t,i}))/(A_{UAB,t,i} + A_{IIB,t,i} + A_{MCB,t,i})$ to ensure consistency between the account balances used for graph generation and the median account balances obtained after optimization.

Once the terms above have been found for the DM and SM product, and noting that for $i \in \{DM, SM\}$ that $D'_{t,i} = D'_{UAB,t,i} + D'_{IIB,t,i} + D'_{MCB,t,i}$, the remaining consumption required to attain the median consumption level (i.e., $\text{med}(C_t) - D'_{t,DM} - D'_{t,SM} - P_{t,x_p}$) is found. Again to ensure consistency between the median optimized results, we attribute this remaining required consumption entirely to the PPA. This means at each time t we set $\text{med}(A_{t,PPA})$ equal to the value that satisfies the equation

$$\text{med}(A_{t,PPA}) = (\text{med}(C_t) - D'_{t,DM} - D'_{t,SM} - P_{t,x_p}) \times \ddot{a}_{x+t,PPA}.$$

Following this adjustment to $\text{med}(A_{t,PPA})$, the values of $A_{UAB,t+1,PPA}$,$A_{IIB,t+1,PPA}$,$A_{MCB,t+1,PPA}$, and the corresponding attribution of drawdowns across $D'_{UAB,t,PPA}$, $D'_{IIB,t,PPA}$, and $D'_{MCB,t,PPA}$ are found by repeating Equations (A.1) through (A.6) whilst setting $i = PPA$. The exception to this is that $A_{MCB,t,PPA} = D'_{MCB,t,PPA} = 0$ for $t \geq 0$. For completeness, we note that $D'_{t,PPA} = D'_{UAB,t,PPA} + D'_{IIB,t,PPA}$.

Finally, for each time t, to obtain the shaded area attributable to 'Account Balance', we evaluate $\sum_i D'_{UAB,t,i}$. To find the shaded area attributable to 'Investment Income' we evaluate $\sum_i D'_{IIB,t,i}$. To find the shaded area attributable to 'Mortality Credits', we evaluate $\sum_i D'_{MCB,t,i}$. Alongside the shaded area for pension income (derived directly from P_{t,x_p}), this generates the graphs presented.

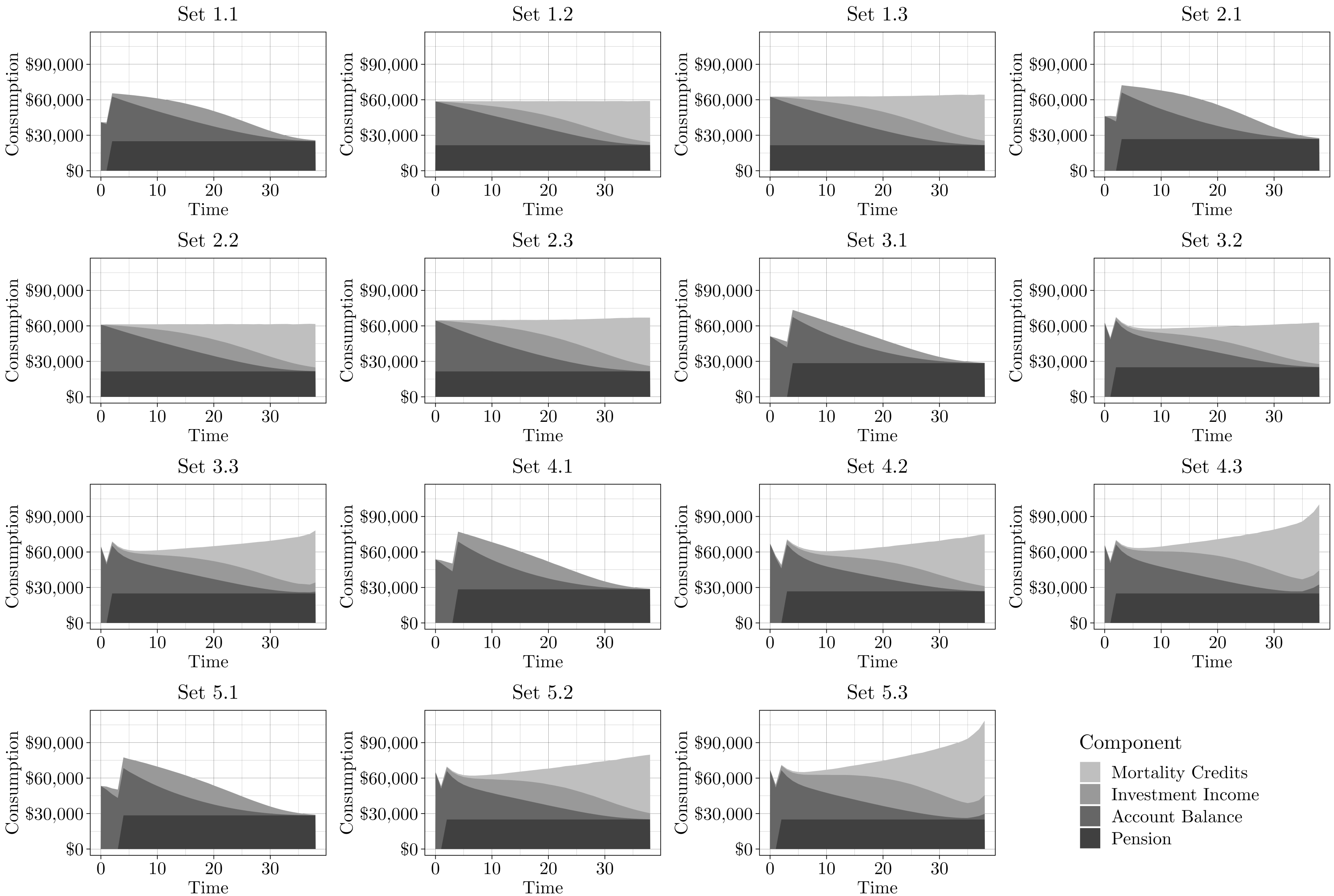

Figure A.1: All median consumption profiles